Analytic Descriptions of a plasma

Microscopic (5tochastic) Description of Charged Many Particle Systems

Let us consider a collection of N charged, "point-size" particles of mass *m* and charge *q* which are free to move as they interact with a self consistent electromagnetic field. The dynamical equations of motion for each particle at stochastic position $\hat{r}_i(t)$ and velocity $\hat{v}_i(t)$ are:

$$\frac{d\hat{r}_i}{dt} = \hat{v}_i \qquad (1a)$$

$$\frac{d\hat{v}_i}{dt} = \frac{q}{m}\left[\hat{E}\left(\hat{r}_i, t\right) + \hat{v}_i \times \hat{H}\left(\hat{r}_i, t\right)\right] \qquad (1b)$$

with i=1,2,…N. $\hat{E}(r,t)$ and $\hat{H}(r,t)$ are the actual (stochastic) electric and magnetic fields (the self fields of the ith particle being excluded) experienced by particles. These fields satisfy Maxwell's equations in a vacuum containing electric currents produced by the N moving charges, viz:

$$\varepsilon_o \frac{\partial \hat{E}}{\partial t} - \nabla \times \hat{H} = \sum_{i=1}^{N} q\hat{v}_i \delta(r - \hat{r}_i) \qquad (1c)$$

$$\nabla \times \hat{E} + \mu_o \frac{\partial \hat{H}}{\partial t} = 0 \qquad (1d)$$

Equations (1) together with the usual supplementary conditions

$$\nabla \cdot \hat{B} = 0 \quad and \quad \nabla \cdot \hat{E} = \frac{1}{\varepsilon_o}\sum_{i=1}^{N} q\delta(r - \hat{r}_i) \qquad (1e)$$

constitute a microscopic (stochastic) system of particle-field equations describing the detailed motion of identical charged particles in the presence of the electromagnetic fields $\hat{E}$ and $\hat{H}$. In Eq. (1b), it is to be noted that the Lorentz force is the only influence on the particles. Other secondary forces due to gravitational interactions, molecular forces, and relativistic effects have been neglected. Some of these can be included by addition of a non-electromagnetic gradient term to Eq. (1b).

Alternatively, one can formulate the microscopic picture solely in terms of field quantities by using the microscopic distribution function

$$\hat{f}(v,r,t) = \sum_{i=1}^{N} \delta(r - \hat{r}_i(t))\, \delta(v - \hat{v}_i(t)) \qquad (2a)$$

which defines the density of particles in r,v phase space at time t., whence

$$N = \iint \hat{f}(v,r,t)\,dv\,dr \qquad (2b)$$

is the total number of particles. Since $\hat{f}$ is a stochastic function in velocity, space and time, it may properly be called a field variable that contains implicitly all the requisite information on the position and velocity $r_i(t)$, $v_i(t)$ of each particle. A field equation for $\hat{f}$ may be inferred upon expanding $\partial \hat{f} / \partial t$ as.,

$$\frac{\partial \hat{f}}{\partial t} = -\sum_{i=1}^{N}\left[\delta(v-\hat{v}_i)\frac{d\hat{r}_i}{dt}\cdot\nabla\delta(r-\hat{r}_i)+\delta(r-\hat{r}_i)\frac{d\hat{v}_i}{dt}\cdot\nabla_v\delta(v-\hat{v}_i)\right] \quad (2c)$$

Rearranging the δ function terms and noting that $\delta(v-v_i)\,\delta(r-r_i) = 0$, except when $v = v_i$ and $r = r_i$, one can use the replacements $dr_i/dt = v_I = v$ and

$dv_i/dt = (q/m)[\hat{E}(r,t) + v \times \hat{H}(r,t)]$ in (2c) to obtain a kinetic equation for $\hat{f}(v,r,t)$ of the form:

$$\left[\frac{\partial}{\partial t} + v\cdot\nabla + \frac{q}{m}(\hat{E} + v\times\hat{H})\cdot\nabla_v\right]\hat{f} = 0 \qquad (3a).$$

The Maxwell Eqs(1c) and (1d) may then be written as:

$$\varepsilon_o\frac{\partial\hat{E}}{\partial t} - \nabla\times\hat{H} - \int_{-\infty}^{\infty} qv\,\hat{f}\,dv = 0 \qquad (3b)$$

$$\nabla\times\hat{E} + \mu_o\frac{\partial\hat{H}}{\partial t} = 0 \qquad (3c)$$

Equations (3), together with the supplementary conditions and appropriate (stochastic) boundary and initial conditions, provide a self-consistent microscopic field description for a charged many particle system. They are called the microscopic Maxwell-Klimontovich field equations and yield results identical to those obtained by other classical methods employing a larger number of equations via Liouville's theorem as used in statistical mechanics.

It should be noted-that Eqs(1) and (3) implicitly assume existence of the spatial and temporal derivatives of the stochastic variables. Existence of these derivatives (and of the Dirac delta function) will be tacitly assumed. A rigorous mathematical treatment of these questions would obscure the classical formulation of the many particle field equations. An implicit requirement on the physical observables described by Eqs(1) is that their derivatives exist; one rarely requires unduly rapid fluctuations since such rapid fluctuations would imply additional interaction terms in Eqs(1) and would vitiate the classical formulation.

Extensions to systems containing a number of different particle types (with, charge and mass distinguished by q^α, m^α *with* $\alpha = 1,2,\ldots m$) can be obtained readily by observing that (1) or (3) hold for each α particle species. Thus one has for each α species containing N^α particles a kinetic distribution

$$\varepsilon_o\frac{\partial\hat{E}}{\partial t} - \nabla\times\hat{H} + \sum_{\alpha=1}^{m}\int_{-\infty}^{\infty} q^\alpha\, v\,\hat{f}^\alpha dv = 0 \qquad (6a)$$

$$\nabla\times\hat{E} + \mu_o\frac{\partial\hat{H}}{\partial t} = 0 \qquad (6b)$$

$$\left[\frac{\partial}{\partial t} + v\cdot\nabla + \frac{q^\alpha}{m^\alpha}(\hat{E} + v\times\hat{B})\cdot\nabla_v\right]\hat{f}^\alpha = 0; \quad \alpha = 1,2\ldots m \quad (6c)$$

To analyze these equations, stochastic fields will be decomposed into ensemble average and fluctuation parts. Thus a typical field

$$\hat{E} = \langle\hat{E}\rangle + \tilde{E} = E + \tilde{E} \qquad (7)$$

is separated into an average E and a fluctuation $\tilde{E}$, with similar decompositions for $\hat{H}$ *and* $\hat{f}^\alpha$. All fluctuation quantities evidently have zero ensemble averages. Use will be made of derivative commutation relations of the form:

$$< \frac{\partial \hat{E}}{\partial t} > = \frac{\partial}{\partial t} < \hat{E} > \quad and \quad < \nabla \times \hat{E} > = \nabla \times < \hat{E} > \qquad (9)$$

From Eqs(9) and relations of the form (7), one finds for the ensemble average of Eqs(6)

$$\varepsilon_o \frac{\partial E}{\partial t} - \nabla \times H + \sum_{\alpha=1}^{m} \int_{-\infty}^{\infty} q^\alpha \, v f^\alpha dv = 0 \qquad (10a)$$

$$\nabla \times E + \mu_o \frac{\partial H}{\partial t} = 0 \qquad (10b)$$

$$\frac{\partial}{\partial t} f^\alpha + v \cdot \nabla f^\alpha + \frac{q^\alpha}{m^\alpha} < (\hat{E} + v \times \hat{B}) \cdot \nabla_v \hat{f}^\alpha > = 0; \quad \alpha = 1,2 \ldots m \quad (10c)$$

Eqs(10a,b) contain only, average quantities E,H, f^α. However, Eq.(10c) is seen to present a difficulty in that it contains the ensemble average of a product of stochastic field variables and thereby introduces a nonlinear coupling of the average and fluctuation fields. With the definitions:

$$F^\alpha = \frac{q_\alpha}{m_\alpha}(E + v \times B) \quad and \quad \tilde{F}^\alpha = \frac{q_\alpha}{m_\alpha}(\tilde{E} + v \times \tilde{B}) \qquad (11)$$

(10c) may be rewritten in the form :

$$\left[\frac{\partial}{\partial t} + v \cdot \nabla + F^\alpha \cdot \nabla_v \right] f^\alpha = - < \tilde{F}^\alpha \cdot \nabla_v \tilde{f}^\alpha >; \quad \alpha = 1,2 \ldots m \quad (12)$$

The right hand term in (12) implies that the ensemble averaged particle density f^α is not constant on an *r,v* trajectory described by;

$$\frac{dr}{dt} = v; \quad \frac{dv}{dt} = \frac{q^\alpha}{m^\alpha}\left[E + v \times B \right] = F^\alpha \qquad (13)$$

The right hand term in (12) furthermore suggests the presence of particle collisions (correlations) along this trajectory and hence is a collision term that couples average (background) and fluctuation components. This collision term accounts for all interactions not covered by the average fields *E* and *H*.

The ensemble averaged Eqs(10a,b) and (12) constitute a kinetic description of a many particle system including particle collisions (correlations) and have a formal similarity to the stochastic Eqs(6) save for the collisional (fluctuation) coupling term. In the absence of the collisions one obtains:

$$\begin{aligned} &\varepsilon_o \frac{\partial E}{\partial t} - \nabla \times H + \sum_{\alpha=1}^{m} \int_{-\infty}^{\infty} q^\alpha \, v f^\alpha dv = 0 \\ &\nabla \times E + \mu_o \frac{\partial H}{\partial t} = 0 \qquad (14) \\ &\left[\frac{\partial}{\partial t} + v \cdot \nabla + \frac{q^\alpha}{m^\alpha}(E + v \times B) \cdot \nabla_v \right] f^\alpha = 0 \end{aligned}$$

Eqs(14) are called the collisionless Maxwell-Vlasov equations, and although similar to the microscopic Eqs(6), they differ in their use of average rather than stochastic variables. Also the boundary and initial conditions on (14) are deterministic. Subtracting the ensemble averaged, Eqs(14) from the stochastic Maxwell-Klimontovich Eqs(6) one obtains the equations governing the fluctuation fields $\tilde{E}, \tilde{H}, \tilde{f}^\alpha$, viz:

$$\varepsilon_o \frac{\partial \tilde{E}}{\partial t} - \nabla \times \tilde{H} + \sum_{\alpha=1}^{m} \int_{-\infty}^{\infty} q^{\alpha} \, v \, \tilde{f}^{\alpha} dv = 0 \qquad (15a)$$

$$\nabla \times \tilde{E} + \mu_o \frac{\partial \tilde{H}}{\partial t} = 0 \qquad (15b)$$

$$\left[\frac{\partial}{\partial t} + v \cdot \nabla + F^{\alpha} \cdot \nabla_v \right] \tilde{f}^{\alpha} = - \tilde{F}^{\alpha} \cdot \nabla_v f^{\alpha} - (\tilde{F}^{\alpha} \cdot \nabla_v \tilde{f}^{\alpha} - < \tilde{F}^{\alpha} \cdot \nabla_v \tilde{f}^{\alpha} >); \quad \alpha = 1,2 \ldots m \quad (15c)$$

One observes that the fluctuating fields (15) are parametrically coupled to the background fields through the right hand term in Eq(15c)., and that the background fields are coupled to the fluctuating fields via the correlation term in Eq(12). Thus the two sets of field equations are strongly coupled and must be solved in a self-consistent fashion. One notes in the decomposition of the overall kinetic equations into background and fluctuation components that the stochastic nature of the problem is still contained in the fluctuation Eqs(15). Hence, the initial and boundary conditions on the fluctuation equations are still difficult to pose, in contrast to those for the average background fields.